Phase operator on $L^2(Q_p)$ and the zeroes of its resolvent

Debashis Ghoshal

School of Physical Sciences
Jawaharlal Nehru University
New Delhi

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This talk is based on a collaboration with Parikshit Dutta
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Appropriately this can be local only in a p-adic metric!

Outline

Motivation

Phase operator

A simple model in statistical mechanics

Phase operators related to the ζ-function

Riemann zeta function: infinite sum and product

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$$\zeta(s) = \prod_{p \in \text{primes}} \frac{1}{(1 - p^{-s})} = \prod_{p \in \text{primes}} \underbrace{\zeta_p(s)}_{\text{local } \zeta\text{-function}}$$

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Riemann proposed to think of s as a *complex* variable and *analytically continued* it as a meromorphic function on the s-plane.

Riemann zeta as partition function

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$$\zeta(s) \sim \mathrm{Tr}_{\mathcal{H}_-}\left(\mathbb{D}^{-s}\right)$$

$\mathbb{D} \sim \prod_p D_{(p)}$: Vladimirov derivative, $\mathcal{H}_- = \bigotimes_p \mathcal{H}_-^{(p)}$, where $\mathcal{H}_-^{(p)} = L^2\left(p^{-1}\mathbb{Z}_p\right)$ is a subspace of square integrable complex valued functions on the p-adic space $\mathbb{Q}_p$ spanned by the Kozyrev wavelets.

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We shall need a phase operator that is "canonically conjugate" (in a limited sense) to the Hamiltonian.

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Classical phase and its quantum analogue: SHO

The spectra of our Hamiltonians are discrete. A familiar example of such a system is the simple harmonic oscillator (SHO) described by the Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$.

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Classical: $(\phi = \omega t)$

$$x = \frac{1}{\sqrt{2}}\left(Ae^{i\phi} + A^* e^{-i\phi}\right)$$

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Correspondence $\sqrt{\omega}Ae^{i\phi} \to \hat{a} \stackrel{?}{=} e^{-i\hat{\phi}}\sqrt{\widehat{N}}$, $\hat{a}^\dagger \stackrel{?}{=} \sqrt{\widehat{N}}e^{i\hat{\phi}}$

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Canonical commutation relation $[\hat{x}, \hat{p}] = i$ implies $\left[\hat{a}, \hat{a}^\dagger\right] = 1$.
Consequently

$$\left[\hat{N}, e^{i\hat{\phi}}\right] = e^{i\hat{\phi}}, \qquad \left[\hat{\phi}, \hat{N}\right] = i \quad \text{(canonically conjugate pair)}$$

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In the number state basis (eigenstates of the number operator $\hat{N}$) the commutator is

$$\langle n| \hat{\phi} |m\rangle = -i\delta_{nm}$$

which is inconsistent. [Susskind-Glogower]

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  Similar issues are faced in defining an operator canonically conjugate to one with a finite (discrete) spectrum.

SU(2) spin j and phase operator I

SU(2) group generated by hermitian operators $\hat{S}_i,\ i = 1, 2, 3$ admit $2j + 1$ (finite) dimensional representations for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$.

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Eigenstates of phase is a unitary transform (discrete Fourier transform) of these states ($B \in \mathbb{R}$ in the following)

$$|\phi_k\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} e^{-imB\phi_k} |m\rangle, \quad \phi_k = \frac{2\pi k}{B(2j+1)}, \quad k = -j, \cdots, j$$

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Eigenvalues ϕ_k defined mod $\frac{2\pi}{B}$.

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Define the phase operator 'conjugate' to $\hat{S}_3$ by spectral decomposition:

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- For $i\beta = \frac{2\pi k'}{B(2j+1)} + \frac{2\pi n}{B}$, where $0 \neq k' = -j, \cdots, j$ and $n \in \mathbb{Z}$ (this means that $i\beta$ is a difference between the phase eigenvalues $\pmod{2\pi/B}$), then $\phi_k - i\beta$ is again an allowed eigenvalue of the phase operator $\pmod{2\pi/B}$.

SU(2) spin j and phase operator III

In the second case adding and subtracting $i\beta$ to the eigenvalue ϕ_k and using the completeness of basis

$$e^{-\beta B\hat{S}_3}\,\hat{\phi}\,e^{\beta B\hat{S}_3} = \hat{\phi} + i\beta$$

(only for $0 \neq \beta = -\frac{2\pi ij}{B(2j+1)}, \cdots, \frac{2\pi ij}{B(2j+1)} \mod 2\pi/B$).

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This *shift covariance relation* [Busch-Grabowski-Lahti] may be rewritten as a commutator

$$[\hat{\phi}, e^{\beta B\hat{S}_3}] = i\beta\, e^{\beta B\hat{S}_3}$$

only at these infinite number of special imaginary values of β.

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Non-interacting spins is an external magnetic field

Take a one-dimensional lattice: at the n-th site there is an SU(2) spin σ_n which takes values in the spin-j representation. These are subjected to a local magnetic field B_n (along the third direction).
If the spins are non-interacting (or the magnetic field is strong compared to the spin-spin interactions) the Hamiltonian is $H = -\sum_n B_n S_3^{(n)}$.
Hence the energy at the n-th site is $E_n = -B_n \sigma_n$.

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Since the inter-spin interactions are negligible, the partition function of the system is the product of the partition functions at each site

$$Z_n = \mathrm{Tr} e^{-\beta H_n} = \sum_{\sigma_n} e^{\beta B_n \sigma_n} = \sum_{m=-j}^{j} e^{\beta B_n m}$$

Fisher zeroes

At special values of the inverse temperature $i\beta = \frac{2\pi m}{B_n(2j+1)} \pmod{\frac{2\pi}{B_n}}$ where $m \in \{-j, \cdots, j\}$ but $m \neq 0$ the partition function vanishes!

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These zeroes of the partition function in the complex β-plane are called *Fisher zeroes*.

Exactly at these values, the resolvent of the exponential of the phase operator

$$\hat{R}[\hat{\phi}](\phi) = \left(1 - e^{-i\phi} e^{i\hat{\phi}}\right)^{-1}$$

has poles (as a function of $z = e^{i\phi}$).

Dressing the spin model

Let us label the sites of the one-dimensional lattice by the first $\mathfrak{p}$ prime numbers $p = 2, 3, 5, \cdots, \mathfrak{p}$: at the p-th site there is a spin-j valued SU(2) spin σ_p. [Spector]

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Let us also shift the zero of the energy so that the Hamiltonian is $H = -\sum_p B_p \left(\hat{S}_3^{(p)} + j\mathbf{1}\right)$. The eigenvalues of $\hat{N}_p = \hat{S}_3^{(p)} + j\mathbf{1}$ are integers $0, 1, \cdots, \mathfrak{n} = 2j + 1$.

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Since the spins are non-interacting, the phase operator $\hat{\phi}_p$ at the p-th site satisfies the shift covariance relation

$$[\hat{\phi}_p, e^{\beta \sum_2^{\mathfrak{p}} B_p \hat{N}_p}] = i\beta e^{\beta \sum_2^{\mathfrak{p}} B_p \hat{N}_p}$$

only for the special values $\beta = \frac{2\pi i k}{B_p(\mathfrak{n}+1)} \pmod{2\pi/B_p}$ with $k = 1, \cdots, \mathfrak{n}$ and $p = 2, \cdots, \mathfrak{p}$.

A special choice of the magnetic field

Let us choose the local magnetic field as $B_p = \ln p$. The partition function is

$$Z(\beta) = \prod_{p=2}^{\mathfrak{p}} \left(\sum_{m_p=0}^{\mathfrak{n}} e^{\beta m_p \ln p} \right) = \prod_{p=2}^{\mathfrak{p}} \frac{1 - p^{\beta(\mathfrak{n}+1)}}{1 - p^{\beta}}$$

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In the thermodynamic limit $\mathfrak{p} \to \infty$ (formal?)

$$Z(\beta) = \lim_{\mathfrak{p}\to\infty} \prod_{p=2}^{\mathfrak{p}} \frac{1 - p^{\beta(\mathfrak{n}+1)}}{1 - p^{\beta}} = \frac{\zeta(-\beta)}{\zeta\left(-(\mathfrak{n}+1)\beta\right)}$$

a ratio of the Riemann zeta functions. (Note: $\mathfrak{n}$ may be *finite*.)

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a ratio of the Riemann zeta functions. (Note: $\mathfrak{n}$ may be *finite*.)
This has the exact same form as the partition functions of a κ-parafermionic primon gas of [Julia] and [Bakas-Bowick] with $\kappa = \mathfrak{n} + 1$ and $s = -\beta$. It would be interesting to try to relate the parafermionic variables to the spin degrees of freedom.

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- The trivial zeroes of $\zeta(-\beta)$ are *not* zeroes of the partition function.

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- There are additional poles (from the zeroes of the ζ-function in the denominator).
- the nontrivial zeroes are the Fisher zeroes in the complex β-plane. They do not have an accumulation point on the real line. Also conjecturally the zeroes of the ζ-function are isolated. So these zeroes are not related to any phase transition—consistent with what is expected physically.

Analyticity of the PF

The *spectrum of the phase operator* is encoded in the trace of its resolvent:

$$\sum_{p=2}^{\mathfrak{p}} \sum_{n_p \in \mathbb{Z}} \sum_{k_p=0}^{\mathfrak{n}} \frac{1}{1 - e^{i\phi_{k_p} + i\frac{2\pi n_p}{\ln p} - i\phi}} - \sum_{p,n_p} \frac{1}{1 - e^{i\frac{2\pi n_p}{\ln p} - i\phi}}$$

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Outline

Motivation

Phase operator

A simple model in statistical mechanics

Phase operators related to the ζ-function

Incommensurate periodicities

We have a phase operator for each site—that is too many. The sum $\sum_p \hat{\phi}_p$ does not have the desired shift covariance relation. (Because the site dependence of the magnetic field $B_p = \ln p$ leads to incommensurate periodicity.)

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Aggregate phase operator φ: on the state $\bigotimes_p |\phi_{p,k_p}\rangle$ it acts as $e^{i\hat{\phi}_p}$ if **one, and only one**, of the eigenvalues $\phi_p \neq 0$, otherwise this operator acts as the identity.

$$e^{i\varphi} = \sum_{p=2}^{\mathfrak{p}} e^{i\hat{\phi}_p} \prod_{q\neq p} \delta_{\phi_q,0} + \frac{2}{n_{\neq 0}(n_{\neq 0}-1)} \sum_{p_1,p_2=1}^{\mathfrak{p}} \prod_{p_1\neq p_2} (1-\delta_{\phi_{p_1},0})(1-\delta_{\phi_{p_2},0})$$

where $n_{\neq 0} = \sum_p (1-\delta_{\phi_p,0})$ is the number of sites where the phase is non-zero. Eigenvalues are in $\bigcup_p U(1)$ (and not in $(U(1))^{\mathfrak{p}}$.

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Alternatively, project on a subspace, in which only one, and exactly one, phase is different from zero and use the total phase operator in this subspace.

Aggregate phase operator

The aggregate phase operator can be shown to satisfy

$$\left[\varphi, \Pi_1 e^{-\beta H} \Pi_1\right] = i\beta\, \Pi_1 e^{-\beta H} \Pi_1$$

which holds in the subspace (to which we project by Π_1) for all $\beta = \frac{2\pi k}{B_p(\mathfrak{n}+1)} \pmod{2\pi/B_p}$ where $k = 1, \cdots, \mathfrak{n}$ and $p = 2, \cdots, \mathfrak{p}$.

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In the limit $\mathfrak{p} \to \infty$, a closure of this subspace to obtain a closed subspace will be necessary.
Except for a pole at $\phi = 0$, the analytic properties of the trace of the resolvent of the aggregate phase operator are related to that of the partition function.

Another proposal: a simple example

In the basis of eigenstates $\{|n\rangle\}$ $(n = 0, 1, \cdots, \mathfrak{n})$ of the number operator $\hat{N}$, define

$$\hat{\Phi} = \sum_{m \neq n} \frac{i\,|m\rangle\langle n|}{m - n}$$

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The commutator holds in a codimension one subspace. E.g., v_ns can be the nontrivial $(\mathfrak{n}+1)$-th roots of unity. [Galindo]

Local ζ_p-function as trace

The local factors $\zeta_p(s) = 1/(1 - p^{-s}) = \sum_{m=0}^{\infty} p^{-sm}$ can be realised as the trace of the generalised Vladimirov derivative $D_{(p)}^{-s}$ in $L^2\left(p^{-1}\mathbb{Z}_p\right) \subset L^2\left(\mathbb{Q}_p\right)$.

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The eigenvalues are labelled by integers $n_{(p)} \in \mathbb{Z}$. Corresponding eigenstates are denoted by $\left|n_{(p)}\right\rangle$. More details in the next talk by Dutta.

Local phase operator on $L^2\left(p^{-1}\mathbb{Z}_p\right)$

For a fixed value of p, the local phase operator $\frac{i}{\ln p}\sum_{n_a^{(p)}\neq n_b^{(p)}}\frac{|\mathbf{n}_a^{(p)}\rangle\langle\mathbf{n}_b^{(p)}|}{(n_a^{(p)}-n_b^{(p)})}$

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For the full space $\otimes_p L^2(p^{-1}\mathbb{Z}_p)$, we use prime factorisation

$$n = \prod_p p^{n_{(p)}} \longleftrightarrow |\mathbf{n}\rangle = \otimes_p |n_{(p)}\rangle$$

These are (orthonormal) eigenfunctions of the Hamiltonian is $H = \sum_p \ln p \, N_{(p)}$

$$H|\mathbf{n}\rangle = \sum_p n_{(p)} \ln p \, |\mathbf{n}\rangle = \ln n \, |\mathbf{n}\rangle$$

Proposed Toeplitz type phase operator

Take a *finite* linear combination of the form $|\mathbf{v}\rangle = \sum_{\mathbf{n}} v_{\mathbf{n}}|\mathbf{n}\rangle$. Further, we require that $v_{\mathbf{n}} \equiv v_{(n_{(2)},n_{(3)},\cdots)} = \prod_p v_{n_{(p)}}$, i.e., the coefficients factorise.

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$$\Phi_{\text{tot}} = \begin{cases} \displaystyle\sum_{\substack{n_a^{(p)},\, n_b^{(p)}=0 \\ \text{not all } n_a^{(p)}=n_b^{(p)}}}^{\mathfrak{n}} \frac{i\left(\bigotimes_{p_a\le\mathfrak{p}} |\mathbf{n}_i^{(p_a)}\rangle\right)\left(\bigotimes_{p_b\le\mathfrak{p}} \langle\mathbf{n}_b^{(p_b)}|\right)}{\sum_{p\le\mathfrak{p}}(n_a^{(p)} - n_b^{(p)})\ln p} & \text{for } n_a^{(p)}, n_b^{(p)} \le \mathfrak{n} \\ \displaystyle\bigotimes_{p>\mathfrak{p}} |n_{(p)} = 0\rangle\langle n_{(p)} = 0| & \text{otherwise} \end{cases}$$

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For $p > \mathfrak{p}$, $|n_{(p)} = 0\rangle$ is the 'vacuum' state in the number representation. In the limit $\mathfrak{p} \to \infty$ this should be a well defined Toeplitz operator on $\otimes_p L^2(p^{-1}\mathbb{Z}_p)$. However, we do not have a proof.

Extension to Dirichlet L-functions

Riemann ζ-function is a part of the family of Dirichlet L-functions, defined as the analytic continuation of the Dirichlet series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \,\in\, \text{primes}} \frac{1}{1-\chi(p)p^{-s}}, \qquad \text{Re}(s) > 1$$

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Kozyrev wavelets are common set of eigenfuctions of all these (mutually commuting) Vladimirov type derivatives. Indeed the operators can be related by suitable unitary transformations Dutta-DG. The rest of the story is very similar—one can repeat all arguments almost *mutatis mutandis*—therefore, we skip the details.

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By suitably choosing the coefficients that determine the subspace, and the parameter α, the phase operators related to the Riemann ζ- and the Dirichlet L-functions can be related by unitary transformations. Dutta-DG.

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Thank you!